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Exploring near-linearities in price–rate of profit trajectories and the concept of effective rank in input–output matrices

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Abstract

In recent years, there has been a resurgence of interest in the controversies surrounding capital theory. At the heart of these debates are the empirically observed near-linearities in the price–rate of profit and wage rate of profit curves. This article posits that these near-linearities can be attributed to the low effective rank property inherent in the economy's system matrices of input–output coefficients. This suggests that a comprehensive representation of how prices evolve in response to changes in income distribution can be achieved with only a few eigenvalues and their respective eigenvectors. Furthermore, this low-dimensional system possesses the capability to capture the majority of distinctive features that characterize the input–output structure of the economy in relation to price movements.

Keywords: Price–rate of profit trajectories, Effective rank, Vertical integration, Eigendecomposition

JEL Classification: B24, B51, C67, D46, D57, E11, E32

1 Introduction

In recent years, research has repeatedly shown the near-linear shape of price–rate of profit (PRP) trajectories and wage–rate of profit (WRP) curves. While PRP trajectories with pronounced curvatures do exist, they are relatively few, and even fewer exhibit a single extremum. We do not a priori rule out the possibility of two extrema in the economically relevant region, that is, for the rate of profit taking on prices from zero to its maximum (see Mariolis and Tsoulfidis 2016a; Shaikh et al. 2022). The explanations offered for these linearities were based on the characteristic distribution of the eigenvalues of the system matrices (see Mariolis and Tsoulfidis 2011; Tsoulfidis 2021, pp. 132–133). Specifically, in the usual dimensions of input–output matrices, the dominant eigenvalue is notably higher (by 40–60%) than the second, followed by the third and a limited number of subsequent eigenvalues, their exact number of which depending on the size of the matrices. The remainder of subdominant eigenvalues form a long tail and paint an exponentially decreasing distribution.

Three hypotheses have been proposed to elucidate the skewed distribution of eigenvalues and the nearly linear patterns observed in PRP and WRP curves:

1. The (nearly) randomly distributed input–output coefficients (Bródy1997; Schefold 2013, 2020 and 2023).
2. The closeness of vertically integrated compositions of capital (VICC) between sectors; that is, the product of sectoral capital–labor ratios times the Leontief inverse (Shaikh 1984, 1998 and Petrović 1991).[1]
3. The low effective-rank or effective dimensionality of the utilized matrices shapes the exponential fall in their eigenvalues, which in turn determines the near-linear features of PRP and WRP curves (Mariolis and Tsoulfidis 2018; Tsoulfidis 2021 and 2022).

The purpose of this study is to examine the extent to which these three hypotheses are consistent with the available evidence and proceed with the less-researched third hypothesis by operationalizing a novel metric of effective rank based on the Shannon entropy and complemented by a similarly motivated metric. It is important to emphasize from the outset that the apparent near linearity of PRP curves does not imply causality running from the marginal productivity of capital to the rate of profit, as in Samuelson's one-commodity world. The lack of causality underscores the incongruity within neoclassical theory as it has been pointed out in the old capital theory controversies and continues to hold in the current developments in capital theory (see Shaikh 2016a, pp. 429–433, Kurz 2020; Kersting and Schefold 2021).

The remainder of the article is structured as follows: Sect. 2 contains the linear classical model of production and elucidates the derivation of PRP and WRP curves. Section 3 critically evaluates the plausibility of the competing explanations for the observed near-linearities. It also introduces the concept of effective rank (or dimensionality) to discern the impact of eigen or singular values on the behavior of the entire economic system. Section 4 embarks on a spectral decomposition of vertically integrated input–output coefficients in an effort to assess the significance of the polynomial terms and the associated with these eigenvalues. Section 5 illustrates the relative importance of each of these polynomial terms by utilizing actual input–output data of the US economy of 15 sectors of the year 2020. This analysis offers a more profound insight into the practicality and reliability of the proposed approach. Section 6 continues the analysis by testing the effective rank metrics utilizing data from the benchmark input–output tables of the years 2007 and 2012 and discusses the findings and the consistency of the two metrics. Section 7 summarizes the key findings and concludes with the idea that there is overfitting of data and that fewer data and dimensions compressed in few sectors would might suffice to capture the characteristic features determining the motion of the economic system.

[1] For recent updates and extensions, see Ferrer-Hernández and Torres-González (2022) and Torres-González (2022).

2 The linear model of production

In a preliminary step, our analysis begins by assuming a single commodity circulating capital linear model of production, whose unit labor values

$$\boldsymbol{\lambda} = \mathbf{l} + \boldsymbol{\lambda}\mathbf{A}, \tag{1}$$

where $\boldsymbol{\lambda} = 1 \times n$ vector of labor values; $\mathbf{l} = 1 \times n$ vector of labor coefficients; $\mathbf{A} = n \times n$ matrix of input–output coefficients, with elements $0 \leq a_{ij} < 1$, which is nonnegative, irreducible, primitive and diagonalizable with distinct eigenvalues. Then unit labor value is the sum of the labor value of inputs $\boldsymbol{\lambda}\mathbf{A}$, plus the living labor $\mathbf{l}$ added by workers.[2] Throughout this article, we denote matrices in boldface uppercase letters, vectors by lowercase boldface letters, and scalar quantities are indicated in italics. Solving Eq. (1) yields

$$\boldsymbol{\lambda} = \mathbf{l}[\mathbf{I} - \mathbf{A}]^{-1}. \tag{2}$$

The prices of production or equivalently, equilibrium prices are given by:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{b}\mathbf{l} + \boldsymbol{\pi}\mathbf{A} + r\boldsymbol{\pi}\mathbf{A}, \tag{3}$$

where r = the economy-wide average rate of profit; $\mathbf{b} = n \times 1$ vector of the basket of goods that workers purchase with their money wage w; $\boldsymbol{\pi} = 1 \times n$ the left-hand only positive eigenvector corresponding to the maximal eigenvalue r^{-1}. Equation (3) states that prices of production, $\boldsymbol{\pi}$, are equal to the sum of labor cost, $\boldsymbol{\pi}\mathbf{b}\mathbf{l}$, and the cost of circulating capital, $\boldsymbol{\pi}\mathbf{A}$, augmented by the profits estimated on the circulating capital, $r\boldsymbol{\pi}\mathbf{A}$. The solution of Eq. (3) will be

$$\boldsymbol{\pi} r^{-1} = \boldsymbol{\pi}\mathbf{A}[\mathbf{I} - \mathbf{A} - \mathbf{b}\mathbf{l}]^{-1}. \tag{4}$$

The left-hand positive eigenvector defined up to multiplication by a scalar and stands for relative prices, which need to be normalized that is to define the appropriate scalar. For this purpose we estimate the standard commodity $\boldsymbol{\sigma}$, that is the right-hand unique positive eigenvector of the matrix the vertically integrated input–output coefficients $\mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{H}$ (see Pasinetti 1977). Thus, we write:

$$\boldsymbol{\sigma} = R\mathbf{H}\boldsymbol{\sigma}, \tag{5}$$

where R is the maximum rate of profit or the reciprocal of the maximal eigenvalue $1/R$ of the matrix $\mathbf{H}$. The so-derived output proportions or standard commodity $\boldsymbol{\sigma}$ is also defined up to a multiplication by a scalar. A meaningfully defined scalar to be used for the scaling of the output proportions of the matrix $\mathbf{H}R$ is defined as follows:

$$\mathbf{s} = \boldsymbol{\sigma}\left(\frac{\mathbf{e}\mathbf{x}}{\boldsymbol{\lambda}\boldsymbol{\sigma}}\right),$$

[2] Tsoulfidis (2021, ch. 7) provides a comprehensive analysis and application of a realistic numerical example involving input–output data of five meaningfully constructed sectors. The discussion delves into the estimation of labor values, prices of production, and the trajectories of the PRP, along with the WRP curves.

where $\mathbf{x}$ is the $n \times 1$ vector of gross output and $\mathbf{e}$ is the $1 \times n$ unit summation vector or market prices, which are by definition are equal to one.[3]

The next step is to fix the relative prices in (4) and labor values in (2) by the above-defined standard commodity $\mathbf{s}$, and derive the normalized row vector of prices of production:

$$\mathbf{p} = \boldsymbol{\pi}\left(\frac{\mathbf{ex}}{\boldsymbol{\pi}\mathbf{s}}\right),$$

and the monetary expression of labor values, defined as

$$\mathbf{v} = \boldsymbol{\lambda}\left(\frac{\mathbf{ex}}{\boldsymbol{\lambda}\mathbf{s}}\right)$$

(see Shaikh 1998). We establish the following equalities

$$\mathbf{ps} = \mathbf{vs} = \mathbf{ex}.$$

That is, the standard sum of prices of production is equal to the standard sum of values and both are equal to the actual sum of output. Thus, since the money wage $w = \mathbf{pb}$, which is equal to the value of basket of commodities purchased by workers, Eq. (3) after some manipulation can be rewritten in terms of normalized prices as follows:

$$\mathbf{p} = w\mathbf{l} + \mathbf{pA} + r\mathbf{pA} \tag{6}$$

and $\mathbf{p}[\mathbf{I} - \mathbf{A}] = w\mathbf{l} + r\mathbf{pA}$ or $\mathbf{p} = w\mathbf{l}[\mathbf{I} - \mathbf{A}]^{-1} + r\mathbf{pA}[\mathbf{I} - \mathbf{A}]^{-1}$ and

$$\mathbf{p} = w\mathbf{v} + r\mathbf{pH}. \tag{7}$$

We post-multiply Eq. (7) by the normalized standard commodity and we get

$$\mathbf{ps} = w\mathbf{vs} + r\mathbf{pHs}.$$

It follows that

$$\mathbf{ps} = w\mathbf{vs} + rR^{-1}\mathbf{ps}.$$

We divide through by $\mathbf{vs} = \mathbf{ps}$ and we end up with $1 = w + rR^{-1}$ and

$$w = 1 - \rho, \tag{8}$$

which solves for the linear wage relative rate of profit curve, where $\rho \equiv rR^{-1}$, with $0 \leq \rho < 1$. By substituting (8) in Eq. (7), we will have

$$\mathbf{p} = (1 - \rho)\mathbf{v}[\mathbf{I} - \mathbf{H}R\rho]^{-1}. \tag{9}$$

Equation (9) is used in the estimation of the price–relative rate of profit trajectories, which become the focus of our study. After all, Sraffa's (1960) emphasis was on the price-feedback effects and the movement of price consequent upon changes in income distribution.

[3] Miller and Blair (2009, p. 42) note that they "redefine the physical units of measurement for each sector to be the amount that can be bought for $1.00; that is, so that the per-unit price for each sector's output is $1.00".

To analyze the precise shape of the WRP curves, we would need access to more comprehensive input–output data and a longer time frame, especially, if our objective is to draw meaningful comparisons related to technological change. Within this specific context, we can derive estimations of the WRP curves from Eq. (6) as outlined below:

$\mathbf{p}[\mathbf{I} - \mathbf{A} - r\mathbf{A}] = \mathrm{w}\mathbf{l}$ solving for $\mathbf{p}$, we get $\mathbf{p} = w\mathbf{l}[\mathbf{I} - \mathbf{A} - r\mathbf{A}]^{-1}$.

We post-multiply by $\mathbf{x}$ both sides of the above and by invoking normalization condition $\mathbf{px} = \mathbf{ex}$, we arrive at the WRP relation:

$$w = \frac{\mathbf{ex}}{\mathbf{l}[\mathbf{I} - \mathbf{A} - \rho R\mathbf{A}]^{-1}\mathbf{x}}. \tag{10}$$

Hence, we may treat the rate of profit, the independent variable, and by assigning to the relative rate of profit prices $\rho = r/R$ starting from zero (corresponding to the maximum wage) until we attain the maximum profit rate (corresponding to zero wage), we generate the WRP curve of the total economy (see Tsoulfidis 2021, chs. 4, 5 and 7). If the price paths exhibit near linearity, it logically follows that the WRP curves would also approximate linearity, rendering the reswitching of techniques (i.e., a shift from a capital-intensive technique to a labor-intensive one and vice versa, as income distribution changes) a remote possibility.

3 The exponentially falling eigenvalues: three hypotheses

In the following discussion, we aim to assess the degree to which each of the three proposed hypotheses aligns with the specific skewed distribution of eigenvalues, which is responsible for the quasi-linear characteristics of PRP trajectories and WRP curves.

3.1 The randomness hypothesis

Let us start by examining the hypothesis of randomness, or more precisely, the near-randomness of the distribution of the input–output coefficients in matrix $\mathbf{A}$. Empirical evidence consistently indicates that the eigenvalues of matrix $\mathbf{A}$ adhere to a skewed distribution. However, it is essential to recognize that not every skewed distribution of eigenvalues necessarily originates from a matrix that is strictly or nearly random. To meet the criteria for such classification, a matrix must satisfy specific conditions. These include ensuring that all its elements are semi-positive and less than one, in addition to maintaining full rank.

As the dimensions of the random matrix increase, a notable phenomenon becomes apparent: the subdominant eigenvalues tend to approach zero, eventually forming an L-shaped distribution. Empirical research on input–output matrices consistently demonstrates that the spectral gap, defined as the difference between the moduli (or absolute values) of the two largest eigenvalues, decreases as the size of the input–output matrix grows (see Mariolis and Tsoulfidis 2011 and 2014, Shaikh et al. 2022).

Moreover, recent studies (Tsoulfidis 2021 and 2022) present evidence challenging the random matrix hypothesis, indicating that it does not withstand the scrutiny of relevant statistical tests. Beyond these findings, more intuitive and systematic explanations exist, particularly concerning the nature of technological change and the associated input–output coefficients. These coefficients, estimated in constant prices, consistently decline over time, as observed in works such as Carter's (1970) study and the research by Tsoulfidis and Tsaliki (2019). Furthermore, the persistent ranking of industries based on their backward, forward, and total linkages provides an additional basis for rejecting the randomness hypothesis, as highlighted in Tsoulfidis and Athanasiadis' (2022). Shaikh (2016) further argued that the random matrix hypothesis primarily applies to the monetary representation of input–output data. In this context, the entries in the columns of matrix **A** signify the relative shares of various inputs in the total output of each industry. However, when expressed in physical terms rather than monetary terms, these matrices do not exhibit the characteristics of random matrices. In the same spirit, Petri (2021) contended that "[…] the coefficient matrices derived from I-O tables are not random. In fact it seems impossible to expect them to be random. Then different 'methods' for the same sector derived from different I-O tables should have unpredictably different coefficients; on the contrary, bread is going to need flour however produced, cars are going to need metals and paint, and so on; and most of the zeros, if input–output tables were sufficiently disaggregated to show them, would coincide". It is not exactly right that the randomness hypothesis requires purely random all elements of matrix **A**. Schefold (2013, p. 10) for example, notes: "It is clear that the elements of actual input–output tables are not strictly random: they are not independent, in that if, for example, a_{ij} is a chemical used in the production of a pharmaceutical product i, the quantity a_{ik} may denote another chemical required in a precise amount". Finally, Shaikh et al. 2022, conducted experiments with input–output coefficients for the US economy, ranging from a 15×15 industry layout to a detailed 403×403 industry representation. Their research revealed that the eigenvalues of matrix **H** follow a Weibull distribution, which does not align with the near-randomness hypothesis.

3.2 The proximity of the VICCs to the economy's average hypothesis

The exponentially falling distribution of eigenvalues is also consistent with the remaining two hypotheses from which the closeness of VICCs of the industries to the economy-wide average is quite appealing to researchers. The idea is that if the VICCs are too close to each other, except for just a few sectors, it follows that the maximal eigenvalue (along with the top subdominant ones) will be crucial for the behavior of the entire economy lending support to the conceptualization of one-commodity world economies.[4] The remainder of eigenvalues will be flocking together at negligibly small values, whose effect will not be felt in the economy.

The trouble with this hypothesis is that the estimation of VICCs depends on equilibrium prices needed for the estimation of the VICCs. In short, there is a circularity issue, which can be hardly overcome unless the estimations are carried out in terms of labor values, that is, the vertically integrated employment coefficients ($\mathbf{l}[\mathbf{I}-\mathbf{A}]^{-1}$) or market prices, although the latter depend on variations of interest (profit) rate or simply by stipulating that all three kinds of prices, despite their differences, nevertheless end up in quite close estimations of the VICCs. However, the question becomes, how can one decide between too different or too similar VICCs? There is no such metric, and the notion of the VICC, although intuitively sound and in the right direction; nonetheless,

[4] The idea is that if the VICC is the same across sectors, it follows that there is one technique of production and production of a single commodity (Samuelson 1962).

requires further qualifications, which are difficult to come by. Thus, it becomes imperative to invoke (if not contrive) a non-price-dependent metric.

3.3 The low effective rank (or dimensionality) hypothesis

The third in line hypothesis, the effective rank, needs to be defined first and then discuss its explanatory content. Roy and Vetterli (2007) are from the first that proposed a continuous or discrete metric, namely the effective rank of a matrix, which serves to quantify the information content within a signal.[5] This metric is estimated from the well-known Shannon (1948) entropy, specifically the spectral entropy index computed from the singular values, denoted as, s, derived from the matrix $\mathbf{H}R$. These singular values are essentially the square roots of eigenvalues of the matrix $\mathbf{H}\prime\mathbf{H}R^2 \text{or} \mathbf{H}\mathbf{H}\prime R^2$ (see Meyer 2001, pp. 411–412). The advantage of singular values over the eigenvalues is that they are always positive and real. The proposed effective rank metric gives us the number of singular vectors that significantly contribute to the signal, and in our case the movement of prices and their distinctive shape. The Shannon entropy index, a key component of this metric, is defined as follows:

$$S = -\sum_{i}^{n} \sigma_i \log \sigma_i, \tag{11}$$

where $\sigma_i\prime$s stand for the standardized singular values of the matrix, whose effective rank we want to estimate, with $i = 1, 2, \ldots, n$. Thus, we have

$\sigma_i = s_i / \sum_i^n s_i$, where $s_i = s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$ are the singular values.

$$0 \leq S \leq \log(n)$$

The exponential of Eq. (11) gives the effective rank (erank) of the matrix, which may be significantly lower than the nominal rank:

$$\text{erank}(\mathbf{H}R) = e^S. \tag{12}$$

In other words, equation (12) denotes the number of singular values necessary to compress an equivalent amount of entropy as the entire matrix. The nature of the studied process invokes the use of common logarithms precisely because the data are in the digit numeral system, and the rank of a matrix is also in digits. The following are properties of the entropy-based metric of rank:

(i) If $s_i = s = 1/n$ then all singular values are the same and so is the influence each of them at the outcome, that is, minimal. The entropy is at a maximum, equal to $\log(n)$ and the effective rank is also at a maximum equal to $e^{\log(n)}$.

(ii) In the limit where a single singular value significantly dominates over the others, which are comparatively much smaller, the entropy tends to decrease and approach zero. Consequently, the effective rank approaches one.

[5] Their metric is inspired by the work of Campbell (1960).

In short, the effective rank metric is a meaningful representation of continuous or in our case discrete rank, which is maximized when the magnitude of the singular values are all equal and minimized when a single singular value dominates over the others.

Equation (12) gives the effective rank or dimensionality of the matrix which may be significantly lower than the nominal rank. The latter might be equal to n, that is, the maximum number of linearly independent columns (or rows) of the matrix under study. The economic meaning of the above formula is the number of sectors required to encompass the same disorder (entropy) as is contained in the total economy. The Shannon entropy is utilized in industrial economics to measure the degree of diversity and concentration and presence of a sort of monopoly power (see Hart 1971 and Tirole 1988, p.222). It is also utilized in studies of income distribution and the input–output structure (Proops 1983, Mariolis and Tsoulfidis 2013). In recent years the same metric includes applications in analytical political economy in identifying random processes (Scharfenaker 2022). In the case of a random nxn matrix, the nominal rank will be n, that is, the number of linearly independent row or column vectors.

A similar, albeit leaning more towards linearity, result is achieved by applying an alternative effective rank metric, which bears distinct similarities with that based on Shannon entropy (for details see Bunea and Xiao 2015):

$$\frac{trace(\mathbf{H}R)}{\sigma_{max}(\mathbf{H}R)}. \tag{13}$$

Since our matrix $\mathbf{H}R$ is diagonalizable, its trace is equivalent to the sum of its eigenvalues weighted by the maximal singular value of the same matrix.

4 Spectral decomposition of the vertically integrated input–output coefficients

Before subjecting the aforementioned formulas to empirical testing, it is essential to grasp the potential information loss resulting from compressing the economy into a significantly smaller number of sectors, as indicated by the effective rank estimations. To address this, we employ an indirect estimation of the effective rank through a spectral or eigen decomposition of the matrix $\mathbf{H}R$, which although not symmetric nevertheless is diagonalizable (Meyer 2001, pp. 514 and 547). The matrix HR can be expressed in its spectral decomposition form, as detailed by Meyer (2001, pp. 517–8) and Mariolis and Tsoulfidis (2018):

$$\mathbf{H}R = \left(\mathbf{y}_1\mathbf{x}_1\right)^{-1}\mathbf{x}_1\mathbf{y}_1 + \lambda_2\left(\mathbf{y}_2\mathbf{x}_2\right)^{-1}\mathbf{x}_2\mathbf{y}_2 + \cdots + \lambda_n\left(\mathbf{y}_n\mathbf{x}_n\right)^{-1}\mathbf{x}_n\mathbf{y}_n, \tag{14}$$

where $\lambda_i, i = 1, 2, ..., n$ stand for the normalized eigenvalues of the matrix $\mathbf{H}$ with the dominant $\lambda_1 = 1$, and $\mathbf{y}$ and $\mathbf{x}$ are the left-hand $1 \times n$ and right-hand $n \times 1$ eigenvectors, respectively. The first or maximal eigenvalue is denoted by$\lambda_1 = 1$, while the second eigenvalue denoted by λ_2 and the remaining or subdominant eigenvalues are denoted byλ_n. Since each of the formed matrices results from the multiplication of two vectors, their respective rank will be equal to one. Adding more terms increases the nominal rank of the resulting matrices according to the number of added terms. It is important to emphasize that the presence of imaginary eigenvalues does not alter Eq. (14).

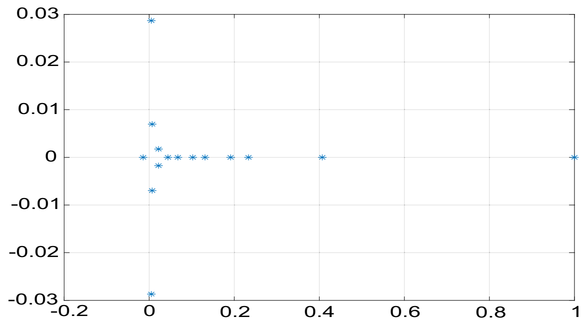

Fig. 1 The location of eigenvalues of matrix **H***R* in the complex plane

Furthermore, in empirical input–output data, the imaginary part of eigenvalues typically appears in the subdominant lower ranks, which are generally much smaller than their also small real part. Therefore, complex numbers need not enter into the approximations through Eq. (14).

The empirical evidence supports our perspective across various countries and over multiple years. Based on our numerical illustration, Fig. 1 visually represents the imaginary part of eigenvalues on the vertical axis and the real part on the horizontal axis. Notably, the imaginary part appears when the real part is exceptionally small. Setting a significance threshold for the eigenvalues of our matrix **H***R* at 0.2, the top three eigenvalues compress the majority of the system's information. Raising the threshold to 0.3 results in only the top two eigenvalues containing most, if not all, necessary information of matrix **H***R*. These eigenvalues and their corresponding eigenvectors are all real and positive. This eigenvalue distribution suggests that the effective rank of our matrix **H***R* falls between 2 and 3. Consequently, the effective interdependence of industries in the economy is significantly smaller than implied by the nominal interdependence or nominal rank. This finding supports the notion of a simpler input–output structure characterizing real economies rather than a complex one. Therefore, a system with just two or at most three eigenvalues is adequate to unveil the key features of an economy's input–output structure concerning price movements. These results hold true across various countries and time spans (see Tsoulfidis 2021, pp. 137–138 and the literature cited there).

It would be of great interest to examine the extent to which the eigendecomposition and effective rank analyses lead to similar results. Furthermore, to explore whether the integration of both methods yields more conclusive insights, particularly from a practical standpoint, regarding the effective dimensions of empirical matrices. To illustrate this, we employ the eigendecomposition and effective rank techniques on a real input–output table of the US economy from the year 2020—the latest available data at the time of writing.

5 An illustrative example based on input–output data of the USA (2020)

The input–output table of the US economy of the year 2020 is available at the 15×15 sectoral structure of total requirements, or what is the same as the Leontief inverse $[\mathbf{I} - \mathbf{A}]^{-1}$. The following are the 15 sectors: 1. Agriculture, etc., 2. Mining, 3. Utilities, 4. Construction, 5. Manufacturing, 6. Wholesale trade, 7. Retail trade, 8. Transportation and warehousing, 9. Information, 10. Finance, insurance, real estate, 11. Professional and business services, 12. Educational services, health care, and social assistance, 13. Arts, entertainment, recreation, accommodation, and food services, 14. Other services, 15. Government.[6] The matrix of input–output coefficients **A** is derived by inverting the Leontief inverse and subtracting it from the identity matrix **I**. Consequently, we obtain the matrix $\mathbf{H} = \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$ representing the vertically integrated input–output coefficients.[7] Furthermore, we calculate the vector of employment coefficients, **l**, by dividing the sectoral wages by the corresponding output available in the commodity-by-the industry table from the same source. This process takes into consideration differences in skills, acknowledging that more skilled workers receive higher wages. In other words, we assume that the issue of skill differences is taken care of by the operation of market forces. We adjust these findings by dividing by the economy-wide average wage, as given by the USA social security administration (https://www.ssa.gov). The workers consumption goods vector, **b**, is determined by multiplying the so-obtained average money wage times the share of personal consumption expenditures of each sector in the total. Utilizing these vectors and matrices, we estimate the actual price paths through Eq. (9) above, where $\rho \equiv r/R$ represents the relative rate of profit. This ratio is the quotient of the rate of profit, denoted as r, assigned to prices ranging from zero to the maximum rate of profit, R. The maximum rate of profit corresponds to the reciprocal maximal eigenvalue of matrix **H**.

For the shake of clarity of presentation and economy in space, Fig. 2 below portrays a panel of eight out of our 15 price trajectories along with their corresponding three approximations—linear, quadratic, and cubic—derived from Eq. (14) above. This selection focuses on eight sectors, specifically those exhibiting the most curved price paths, with the exception of the last one, which follows an almost linear trajectory and is included to complete the panel of graphs in Fig. 2 for illustrative purposes. The horizontal axis of each graph represents the relative rate of profit, ρ, while the vertical axis depicts the ratio of the estimated price, **p**, over the direct price, **v**, denoted as p_i/v_i. The straight green lines correspond to linear approximations of estimated price trajectories, as determined by Eq. (9), with ρ values ranging from zero to nearly 1. The black dashed lines represent square approximations, the red dots signify cubic approximations, and the blue with dots line represents the actual estimated prices whose paths we seek to approximate. When the estimated price of production surpasses the direct price ($p_i/v_i = 1$), the industry is deemed capital-intensive; conversely, a labor-intensive industry exhibits a ratio below 1. A crossing of the $p_i/v_i = 1$ line from above or below signifies a change in the characterization of an industry's capital intensity, from capital-intensive to labor-intensive or vice versa.

[6] The data are available at the USA Bureau of Economic Analysis (BEA) (2023), whose link is the following: BEA: Previously Published Estimates. We select SUPPLY-USE and the input–-output total requirements data for the year 2020 can be downloaded from the filename: IXI_TR_1997-2021_PRO-SEC.

[7] The market prices are supposed to be equal to one monetary unit, whatever this happens to be. In other words, because input–output tables are constructed in terms of aggregated industries, there is no physical measure of the sectoral output. Consequently, the adoption of a unit of measurement, such as one million USD worth of sectoral output becomes compelling. This convention yields equivalent results to expressing all values in physical measurement units. For further elaboration and numerical illustrations, see Miller and Blair (2009, ch. 2).

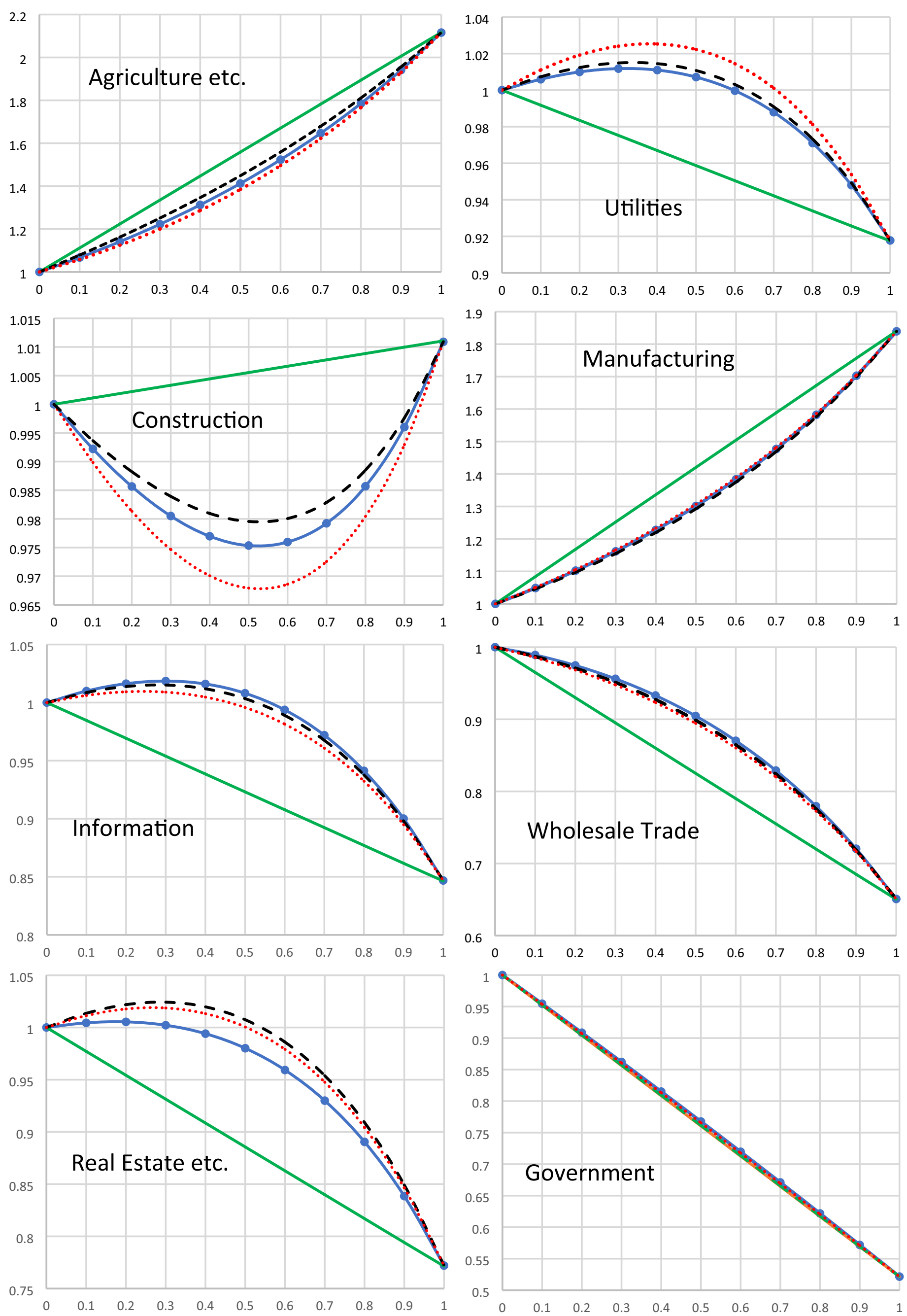

Fig. 2 Price trajectories and their linear, quadratic and cubic approximations

We observe that the linear approximation (as this can be judged by the mean absolute deviation) is quite satisfactory even in case that one would only accept a relatively minimal deviation. The quadratic approximation, in general, is an improvement over the linear, even for this small size of input–output structure. However, the same cannot be asserted with the cubic, which we find, in most cases, excessive and therefore redundant.

One would be wondering if the same answer regarding the number of terms or the rank suggested by the eigendecomposition would be derived through the application of

Table 1 Singular values, Shannon's entropy and effective rank[a]

Ranking of singular values	Singular values (1)	Normalized singular values (2)	Common logarithms of (2) (3)	The product of (2)x(3) (4)
1	1.439	0.477	−0.322	−0.153
2	0.569	0.188	−0.725	−0.137
3	0.287	0.095	−1.022	−0.097
4	0.195	0.064	−1.191	−0.077
5	0.158	0.052	−1.282	−0.067
6	0.110	0.036	−1.439	−0.052
7	0.083	0.027	−1.561	−0.043
8	0.052	0.017	−1.764	−0.030
9	0.040	0.013	−1.873	−0.025
10	0.028	0.009	−2.028	−0.019
11	0.019	0.006	−2.192	−0.014
12	0.017	0.006	−2.237	−0.013
13	0.010	0.003	−2.487	−0.008
14	0.007	0.002	−2.619	−0.006
15	0.004	0.001	−2.846	−0.004
Sum:	3.020	1.000	Shannon (S)	0.746
			erank $= e^{S}$	2.107

[a] The effective rank remains the same when employing natural (instead of common) logarithms, given the appropriate adjustment of the base. Consequently, the Shannon entropy S in terms of natural logarithms reformulates relation (12) as$e^{S/\ln(10)}$

the exponential of the Shannon index of entropy. For this purpose, we estimate the singular values, s_i of the matrix $\mathbf{H}R$ or what is the same the square roots of the eigenvalues of the matrix $\mathbf{HH}'R^2 = \mathbf{H}'\mathbf{H}R^2$. Our estimates are shown in Table 1 below:

From Shannon entropy, denoted as S, with an exponential value of 2.107, the proximity of the effective rank of the system matrix to two implies that utilizing only the first two terms of Eq. (14) would yield a highly accurate approximation of the actual price path. As mentioned above the effective rank is a metric to signify the fact that some industries (columns of the matrix) are more fundamental than others, in that the other industries have a structure which is *almost* a linear combination of the fundamental sectors. In such cases, it is anticipated that the effective dimensionality of the matrix will be considerably lower than its nominal dimension. The effective rank metric is designed to capture this proximity, providing a measure that is generally not an integer, though it will eventually be rounded. This approach acknowledges that the effective rank metric, borrowed from probability and/or information theory, lacks the precision of its counterpart in linear algebra. Consequently, one should anticipate a non-integer result, necessitating rounding for meaningful comparison with the standard algebraic rank of the matrix.

As expected, a comparable estimate arose from the utilization of the alternative effective rank metric outlined in relation (13). Thus, we got:

$$\frac{trace(\mathbf{H}R)}{\sigma_{max}(\mathbf{H}R)} = \frac{2.2305}{1.4392} = 1.5498$$

which is consistent with the approximations derived from our spectral decomposition, where the cubic factor exhibited only marginal improvements This suggests that there

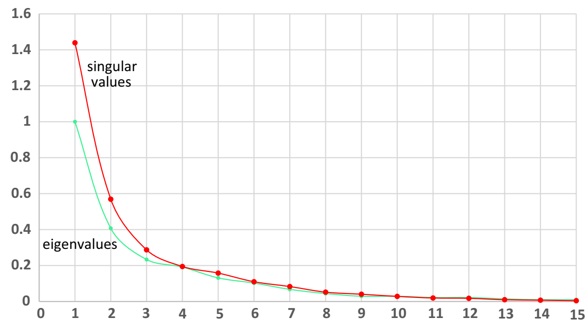

Fig. 3 Absolute eigenvalues and singular values of the matrix **H***R*, USA 2020

may be no need to incorporate the cubic term in the factorization of the 15 × 15 size input–output structure. Figure 3 below provides a visual comparison of both scenarios.

6 Results on the effective rank and their evaluation

We have conducted experiments using input–output data of varying dimensions and spanning multiple years of the US economy. Specifically, our analyses of the benchmark input–output data for the years 2007 and 2012 indicate that a quadratic approximation is suitable for dimensions of 15 × 15. For higher dimensions, such as the 71 × 71 industries input–output matrices, the effective rank was nearly twice higher than that of the 15 × 15 dimensions, but when we consider the difference in dimensions we find that the effective rank changed only marginally. A result, confirmed by the spectral decomposition which showed that for all practical purposes, a cubic term is more than enough for a satisfactory approximation. Further attempts with the fourth or fifth term did not enhance the accuracy of the approximation (Tsoulfidis 2022). We also examined the input–output data with dimensions of 405 × 405 for the benchmark years 2007 and 2012. The effective ranks for these matrices were found to be 8.648 and 8.495, respectively (see Fig. 4, below). However, eigen approximations were not explored for these exceptionally high-dimensional input–output tables.

In examining matrices of lower dimensions, specifically the 54 × 54 matrices for the USA in 2007 and 2012 (Timmer et al. 2015), our findings remained consistent. In both instances, the quadratic approximation of price trajectories proved highly satisfactory (Tsoulfidis 2021). Interestingly, the inclusion of cubic and quartic terms did not enhance the accuracy of the approximation, even for prices of production with trajectories characterized by the more pronounced curvature. Remarkably, these price of production trajectories exhibit surprising small deviation from direct prices, implying a close proximity of their VICCs to the economy-wide average ones or, equivalently, the Sraffian standard ratio.

The effective rank, determined by the ratio of the trace to the maximal singular value of matrix **H***R*, is highlighted for specific years with starred markers. Upon a cursory consideration of Fig. 4, it becomes apparent that, as the economy undergoes increased disaggregation, the effective rank exhibits a slight elevation in comparison to the expansion in the dimensions of the matrix. This observation challenges the (near-) randomness hypothesis, according to which, as the dimensions of a matrix expand towards infinity,

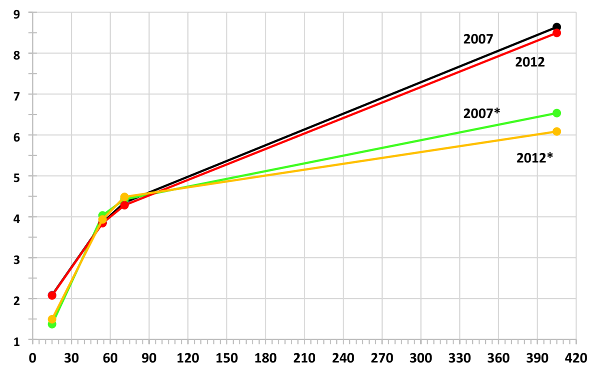

Fig. 4 Effective ranks for matrices of different dimensions, USA, 2007 and 2012

the system's stochastic nature should manifest, leading to a reduction in rank and eventual convergence to 1. It becomes abundantly clear that neither the so-called spectral gap (the difference between the moduli, or absolute values, of the two largest eigenvalues) increases nor the respective estimations of the effective rank confirm such a hypothesis, for a reasonable (economically meaningful) increase in the size of input–output matrices. Moreover, the observed increase in effective rank with the expansion of dimensions suggests an exponentially falling distribution of the eigenvalues, resembling the findings of Shaikh et al. (2022) that align with a Weibull distribution.

Both metrics for effective rank yield similar results for matrices of intermediate size. Nevertheless, as the number of sectors expands, the Shannon entropic definition metric shows a considerable and swift increase, likely due to its exponential nature. In contrast, the trace to the maximal singular value metric produces comparable results. The effective ranks are lower for the 15×15 industry structure, approximately the same for intermediate-sized (54×54 and 71×71) input–output structures, and somewhat lower for the matrices of the more detailed 405×405 industry structure.[8] These findings lend support to the view that as the dimensions of the matrix $\mathbf{A} \text{or} \mathbf{H}$ increase, there is corresponding increase in the number of subdominant eigenvalues or singular values, which become influential for the motion of the economy. Furthermore, by taking the ratio of the entropy-based effective rank to its maximal effective rank, we estimate the percentage of compressed information to the total available, that is, the ratio:

$$\frac{e^S}{e^{\log(n)}} \text{or} e^{S-\log(n)},$$

which for the 15×15 matrix of 2020 gives 0.65, that at least 65% of the total information contained in the matrix $\mathbf{H}R$ is compressed in the first two singular values. Similar are the entropy-based effective rank results for the years 2007 and 2012 for the 15×15, 54×54, 71×71 and 405×405 dimensions which we display in Table 2 below.

In the SVD (singular value decomposition) approximation, the requirement to distinguish the dominant from the subdominant singular values is to acquire nearly 2/3 of the

[8] The data for the 54×54 input–output structure come from the World Input–Output Database (Timmer et al. 2015) and the link for the data is wiod@rug.nl. The input–output data for the US economy are from the Bureau of Economic Analysis and refer to industry-by-industry total requirements. For the link see footnote 6.

Table 2 Effective rank and percentage of compressed information

Years	Dimensions			
	15 × 15(%)	**54 × 54(%)**	**71 × 71(%)**	**405 × 405 (%)**
2007	64.3	68.2	68.3	63.7
2012	64.0	68.0	67.3	65.3

total, which is a usual parsimonious threshold to capture the most important features of the original matrix while minimizing the amount of error or information loss. Similarly, the measure of the effective rank provides valuable insights into the structure and complexity of the underlying data and reveals how many of the singular values of the matrix $\mathbf{H}R$ are needed to acquire the nearly 66.6% of the economic system's matrix information, a percentage attained by a surprisingly small number of singular values.

The results for the countries that we tested were no different from those of the US economy. The distribution of eigen and singular values exhibited a consistent pattern described by the same exponential equation, which fit well with the distribution of eigenvalues across all years and countries tested (Tsoulfidis 2021). These findings lead to the idea that there are certain regularities embedded deeply in the available input–output data, and they are manifested through the skew distribution of eigen or singular values, which are in turn determined by the effective rank (or dimensions) of the system matrices.

From a mathematical perspective, the concept of effective rank and its related dimensions, estimated through Shannon entropy or trace indexes, is well-founded. This is primarily because the top singular values demonstrate notable distinctions from the rest, containing a considerably greater amount of explanatory information. Our analysis of the 15×15 input–output structure indicates an effective rank of two when rounded to the nearest integer, a result consistent with that obtained through eigendecomposition. In practical terms, it is feasible to construct an analytical model featuring just two or a few pivotal sectors that encapsulate the core dynamics of the overall economy, particularly regarding price movements. Therefore, it is appropriate to designate these sectors as the primary drivers of the economy, granting them more substantial significance in a simple yet comprehensive economic model. It is worth reiterating that as the number of sectors increases, in the empirical matrices we have hitherto examined, we observe that the effective rank of these matrices increases, albeit at a rate far lower than the number of industries. This finding suggests that while additional industries become more relevant in the economic landscape, their number remains limited and conditioned by the percentage of compressed information they contain.

Finally, the matrix of fixed capital stock, derived through the capital flow tables, indicated much lower dimensions, and the quadratic term would be more than enough.[9] After all, the second eigen- or singular value in these matrices is markedly lower than the

[9] Only a few matrices of capital (or investment) flows are available. The latter, with proper multiplication by the inverse of the diagonal matrix of capital stock per unit of output gives rise to the matrix of capital stock coefficients (see Tsoulfidis 2021, ch. 7). For instance, the latest capital flows matrix of the USA is that of the year 1997 of the 65×65 structure. The capital flow tables are rarely published, so matrices of capital stock for the missing years can be constructed based on restrictive but not unrealistic assumptions.

maximal. Besides, in capital stock matrices, as expected, there are too many rows with zero elements. The idea is that neither the consumer goods industries nor services produce capital goods, so their rows are filled either by zeros or relatively small numbers. It is important to point out that the post-multiplication of the capital stock matrix by the Leontief inverse gives rise to a new matrix, whose form takes on that of the capital stock matrix. In counting the number of zeros in our 65×65 capital stock matrix, we found 39 rows that added to the zeros scattered to the rest of the cells amounted to 61 percent of the total figures of the capital stock matrix, without counting the near-zero negligibly small elements (Tsoulfidis 2021, pp.71–78 and 181). These findings suggest that the more concrete the analysis by including the capital stock matrix, $\mathbf{K}$, the sparser the resulting $\mathbf{K}[\mathbf{I}-\mathbf{A}]^{-1}R$ matrices, the lower their nominal rank and so the effective rank becomes and in fact is found much lower than that estimated in the respective circulating capital model. Consequently, the more the L-shaped distribution of eigenvalues, the more nearly linear the price and WRP curves. A result well-established in the pertinent literature (see Mariolis and Tsoulfidis 2011 and 2016b, Shaikh 2016, p. 441–2, Tsoulfidis 2021). In terms of the spectral effective rank and compressed information, we can tell that the structure of the economy becomes even simpler and therefore much easier approximated through the use of a couple of terms or singular values.

From the above, we deduce that the distributions of eigen- and singular values decrease exponentially fast, toward zero. This result is repeatedly found in the US economy regardless of the number of industries and over different spans of time. The same pattern has also been found in the data of all hitherto tested input–output data and countries over the years. Thus, rightfully, this distribution of eigenvalues may take the characterization of a 'stylized fact', the result of the low effective rank of the system matrices. The economic meaning of this finding is that only for very few industries do their input–output coefficients change independently of the rest of the input–output structure of the economy. The remainder of the industries depends on developments taking place in these rather few but, in a sense, hyper-industries that deserve further investigation.

7 Summary and concluding remarks

From our discussion, it follows that both the spectral decomposition and the effective rank metrics complement each other and separate and combined contribute to our approximation of economic reality, as described in its input–output structure. The information derived supports the idea that the complex structure of the economy can be distilled to a few key sectors, simplifying the complexity inherent in economic systems. The hitherto analysis has shown that Samuelson's (1962) one-commodity world description of the economy was an oversimplification, but so was Ricardo's corn model. Marx's schemes of simple reproduction could also be taken as a one-commodity world because of the assumption of the equal organic composition of capital between departments. In this line of research, we can also treat Sraffa's standard system and the device of the standard commodity. All these lend support to the idea of simpler structures, within which may be compressed most of the information needed to explain the motion of the economy.

Our findings of near-linear price trajectories by no means suggest that the neoclassical theory is consistent in dealing with real-world features. On the contrary, the problems of the marginal productivity theory of income distribution persist, as the equality of marginal productivity of a factor production with its payment results from an identity, and not from a causal relationship from the marginal product of a factor to the rate of its payments as expected in the neoclassical theory (see Shaikh 2016, ch. 9). Furthermore, the assumption of given endowments characterized by high substitutability and the subjective nature of preferences permeate the neoclassical analysis not only in its pure exchange description, but also in its models of production with produced means of production. We have shown that for the usual input–output structure of the economy, the first couple of eigen or singular values are adequate for the construction of models that mimic the operation of the entire economy to questions of price movements, among others. In this respect, the principal components analytical method may be used, and it has been used effectively in this direction (Tsoulfidis and Athanasiadis 2022).

In short, the applied eigen (or spectral) decomposition method revealed that the input–output structure of the economies is simpler than is usually thought, and a lot of information is compressed in the maximal eigenvalue of the system matrices while the remaining eigen- or rather singular-values add little practically useful additional information. Thus, by limiting ourselves to the first few terms of the eigendecomposition, we obtain a satisfactory approximation of the price trajectories consequent upon changes in income distribution. In so doing, we end up with the view that the actual economies are not like a one-commodity world. The latter would require equal capital intensities between industries, which is another way to say that the system's matrices would have nominal and effective rank equal to one. This does not mean that our multi-commodity world requires all produced commodities and dimensions to uncover its structural features. From the above it follows that it is quite reasonable to assume that a two-sectoral (akin to the Marx–Feldman–Mahalanobis 'corn-tractor' model) or a three-sectoral model would be sufficient as a realistic substitute for an actual disaggregated economy and fully explain the price trajectories (and their difference from labor values).

Based on our research, which focuses on effective rank, we have identified a limited number of industries that consistently stand out among the diverse array of industries over time. Our analysis began by examining the input–output structure of 15 industries in the year 2020 and subsequently extended to include benchmark years 2007 and 2012, exploring various industry structures. The initial 15×15 industry framework evolved progressively into larger structures, namely 54×54, 71×71, and ultimately, a comprehensive 405×405 industry structure. Our empirical findings reveal that as the number of industries expands, the effective rank experiences only marginal increases. This suggests that a specific set of industries, typically numbering from a minimum of two to a maximum of nine, depending on the level of industry detail, play a pivotal role. Furthermore, these key industries seem to maintain consistency over the years and act as carriers of technological changes. In contrast, the remaining industries exhibit a nearly linear dependence on these key industries.

Thus it comes as no surprise that the Shannon's metric and eigendecomposition are related to the opinion expressed by Mariolis and Tsoulfidis (2016a, b), Tsoulfidis (2021) Petri (2021) and Shaikh et al. (2022) about the structure of linear models of production that is, that the interdependence and therefore the structure of the industries is not random and thus a shift in the distribution of technical coefficients (indicating a technological change) cannot happen randomly. Consequently, our results may give a reason as of why dimensionality reduction is both possible and desirable within a deterministic modern classical context. In a nutshell, we are dealing with over fitting data and overdimensional representations of the actual economies. Our analysis lends support to the view that the deep laws of motion of the system can be laid bare by de-noising our data and meaningfully compressing the dimensions of the system to just a few.

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Availability of data and materials
The datasets for the present study are available electronically and publicly. Data on input–output tables are available from the website of the USA Bureau of Economic Analysis (BEA).

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The author declares that they have no competing interests.

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