The Samuelson macroeconomic model as a singular linear matrix difference equation

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Abstract

In this paper, we revisit the famous classical Samuelson's multiplier–accelerator model for national economy. We reform this model into a singular discrete time system and study its solutions. The advantage of this study gives a better understanding of the structure of the model and more deep and elegant results.

Keywords: Samuelson, Macroeconomic, Singular, System, Difference equations

1 Introduction

Many authors have studied generalized discrete, see (Apostolopoulos and Ortega 2019; Dassios 2015a; Dassios and Kalogeropoulos 2013; Ogata 1987; Ortega and Apostolopoulos 2018), and continuous time systems, see (Dassios et al. 2019, 2020; Lewis 1986, 1987, 1992; Liu et al. 2019; Milano and Dassios 2016, 2017), and their applications especially in cases where the memory effect is needed including generalized discrete, see (Dassios and Baleanu 2015; Dassios and Kalogeropoulos 2014; Dassios and Szajowski 2016), and continuous time systems with delays, see (Liu et al. 2017, 2019, 2020; Tzounas et al. 2020). Many of these results have already been extended to systems of differential, see (Dassios and Baleanu (2018a, b; Klamka 2010; Klamka and Wyrwał 2008; Podlubny 1999) and difference equations with fractional operators, see (Dassios 2015c, 2018b).

Keynesian macroeconomics inspired the seminal work of (Samuelson 1939), who actually introduced the business cycle theory. Although primitive and using only the demand point of view, the Samuelson's prospect still provides an excellent insight into the problem and justification of business cycles appearing in national economies. In the past decades, many more sophisticated models have been proposed by other researchers, see (Barros and Ortega 2019; Dassios and Devine 2016; Dassios and Zimbidis 2014; Dassios et al. 2014; Dorf 1983; Kuo 1996; Puu et al. 2004; Rosser 2000). All these models use superior and more delicate mechanisms involving monetary aspects, inventory issues, business expectation, borrowing constraints, welfare gains and multi-country consumption correlations.

Some of the previous articles also contribute to the discussion for the inadequacies of Samuelson's model. The basic shortcoming of the original model is the incapability to produce a stable path for the national income when realistic values for the different

parameters (multiplier and accelerator parameters) are entered into the system of equations. Of course, this statement contradicts with the empirical evidence which supports temporary or long-lasting business cycles.

In this article, we propose an alternative view of the model by reforming it into a singular discrete time system.

The paper theory of difference equations is organized as follows. Section 2 provides a short review for the organization of the original model and in Sect. 3, we introduce the proposed reformulation into a system of difference equations. Section 4 investigates the solutions of the proposed system.

2 The original model

The original version of Samuelson's multiplier–accelerator original model is based on the following assumptions:

Assumption 2.1   National income T_k in year k equals to the summation of three elements: consumption, C_k, private investment, I_k, and governmental expenditure G_k

$$T_k = C_k + I_k + G_k. \tag{1}$$

Assumption 2.2   Consumption C_k in year k depends on past income (only on last year's value) and on marginal tendency to consume, modeled with a, the multiplier parameter, where $0 < a < 1$,

$$C_k = aT_{k-1}. \tag{2}$$

Assumption 2.3   Private investment I_k in year k depends on consumption changes and on the accelerator factor b, where $b > 0$. Consequently, I_k depends on national income changes,

$$I_k = b(C_k - C_{k-1}) = ab(T_{k-1} - T_{k-2}). \tag{3}$$

Assumption 2.4   Governmental expenditure G_k in year k remains constant

$$G_k = \bar{G}.$$

Hence, the national income is determined via the following second-order linear difference equation

$$T_{k+2} - a(1 + b)T_{k+1} + abT_k = \bar{G}.$$

See (Samuelson 1939) for the needed theory of difference equations that lead to the solution of the above equation.

3 The reformulation: Singular Samuelson's model

Let

$$Y_k = \begin{bmatrix} T_k \\ C_k \\ I_k \end{bmatrix}.$$

Then, (1) can be written as

$$0 = -T_k + C_k + I_k + G_k,$$

or, equivalently,

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} Y_{k+1} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} Y_k + G_k.$$

Equation (2) can be written as

$$C_{k+1} = aT_k$$

or, equivalently,

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} Y_{k+1} = \begin{bmatrix} a & 0 & 0 \end{bmatrix} Y_k.$$

Finally (3) can be written as

$$I_{k+1} = b(C_{k+1} - C_k).$$

or, equivalently,

$$-bC_{k+1} + I_{k+1} = -bC_k.$$

or, equivalently,

$$\begin{bmatrix} 0 & -b & 1 \end{bmatrix} Y_{k+1} = \begin{bmatrix} 0 & -b & 0 \end{bmatrix} Y_k.$$

Hence the above expressions can be written in a matrix form. We proved the following theorem:

Theorem 3.1 *The difference equation* (1) *can be written in the form of the following singular discrete time system:*

$$FY_{k+1} = GY_k + V_k, \quad k = 2, 3, \dots, \tag{4}$$

where

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -b & 1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 1 & 1 \\ a & 0 & 0 \\ 0 & -b & 0 \end{bmatrix}, \quad V_k = \begin{bmatrix} G_k \\ 0 \\ 0 \end{bmatrix}.$$

Note that F is singular (detF $= 0$). Throughout the paper, we will use in several parts matrix pencil theory to establish our results. A matrix pencil is a family of matrices $sF - G$, parametrized by a complex number s, see (Dassios and Baleanu 2013).

Definition 3.1 Given $F, G \in \mathbb{R}^{r \times m}$ and an arbitrary $s \in \mathbb{C}$, the matrix pencil $sF - G$ is called

1. Regular when $r = m$ and $\det(sF - G) \neq 0$;
2. Singular when $r \neq m$ or $r = m$ and $\det(sF - G) \equiv 0$.

Corollary 3.1 *The system* (4) *has always a regular pencil* $\forall a, b$.

Proof The determinant $\det(sF - G) = s^2 - a(b+1)s + ab \neq 0$. Hence from Definition 2.1, the pencil is regular. The proof is completed. ∎

The class of $sF - G$ is characterized by a uniquely defined element, known as the Weierstrass canonical form, see (Dassios 2017), specified by the complete set of invariants of $sF - G$. This is the set of elementary divisors of type $(s - a_j)^{p_j}$, called *finite elementary divisors*, where a_j is a finite eigenvalue of algebraic multiplicity p_j ($1 \leq j \leq \nu$), and the set of elementary divisors of type $\hat{s}^q = \frac{1}{s^q}$, called *infinite elementary divisors*, where q is the algebraic multiplicity of the infinite eigenvalue. $\sum_{j=1}^{\nu} p_j = p$ and $p + q = m$.

From the regularity of $sF - G$, there exist non-singular matrices $P, Q \in \mathbb{R}^{m \times m}$ such that

$$
\begin{aligned}
PFQ &= \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix}, \\
PGQ &= \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}.
\end{aligned}
\tag{5}
$$

J_p, H_q are appropriate matrices with H_q a nilpotent matrix with index q_*, J_p a Jordan matrix and $p + q = m$. With $0_{q,p}$ we denote the zero matrix of $q \times p$. The matrix Q can be written as

$$
Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}.
\tag{6}
$$

$Q_p \in \mathbb{R}^{m \times p}$ and $Q_q \in \mathbb{R}^{m \times q}$. The matrix P can be written as

$$
P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.
\tag{7}
$$

$P_1 \in \mathbb{R}^{p \times r}$ and $P_2 \in \mathbb{R}^{q \times r}$.

The solution of system (4) is given by the following Theorem:

Theorem 3.2 (See Dassios 2012) *We consider the system* (4). *Since its pencil is always regular, its solution exists and for $k \geq 0$, is given by the formula*

$$
Y_k = Q_p J_p^k C + Q D_k.
$$

Here, $D_k = \begin{bmatrix} \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i \\ sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i} \end{bmatrix}$ *and $C \in \mathbb{R}^p$ is a constant vector. The matrices Q_p, Q_q, P_1, P_2, J_p, H_q are defined by* (5), (6) *and* (7).

4 Results and discussion

In this section we will present our main results. We will provide the solution to the system (4) and consequently we will derive the sequence for the national income, the consumption and the private investment.

Theorem 4.1 *We consider the system* (4). *Then in year* k, *national income* T_k, *consumption* C_k *and private investment* I_k *are given by*

$$T_k = s_1^{k+1}c_1 + s_2^{k+1}c_2 + a\sum_{i=0}^{k-1}[(s_1^{k-1} + s_2^{k-1})]G_i,$$

$$C_k = a(s_1^k c_1 + s_2^k c_2) + a^2\sum_{i=0}^{k-1}[(s_1^{k-i-1} + s_2^{k-i-1})]G_i,$$

$$I_k = s_1^k(s_1 - a)c_1 + s_2^k(s_2 - a)c_2 + a\sum_{i=0}^{k-1}[((s_1 - a)s_1^{k-1} + (s_2 - a)s_2^{k-1})]G_i.$$

Proof From Corollary 3.1, the pencil $sF - G$ is always regular. Furthermore, the pencil has one infinite eigenvalue and two finite:

$$s_1 = \frac{a(1+b) + \sqrt{a^2(1+b)^2 - 4ab}}{2}, \quad s_2 = \frac{a(1+b) - \sqrt{a^2(1+b)^2 - 4ab}}{2}.$$

From Theorem 3.2, the solution of (4) is given by

$$Y_k = Q_p J_p^k C + Q\begin{bmatrix} \sum_{i=0}^{k-1} J_p^{k-i-1}P_1 V_i \\ sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i} \end{bmatrix}.$$

Since we have one infinite eigenvalue, we have

$$H_q = 0$$

and J_p is the Jordan matrix of the two finite eigenvalues:

$$Y_k = Q_p\begin{bmatrix} s_1^k & 0 \\ 0 & s_2^k \end{bmatrix}C + Q\begin{bmatrix} \sum_{i=0}^{k-1} J_p^{k-i-1}P_1 V_i \\ 0 \end{bmatrix}.$$

The matrix Q_p has the two eigenvectors of the two finite eigenvalues:

$$Q_p = \begin{bmatrix} s_1 & s_2 \\ a & a \\ s_1 - a & s_2 - a \end{bmatrix},$$

while Q_q is the eigenvector of the infinite eigenvalue:

$$Q_q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence,

$$Q = \begin{bmatrix} s_1 & s_2 & 1 \\ a & a & 0 \\ s_1 - a & s_2 - a & 0 \end{bmatrix}$$

and the solution Y_k takes the form:

$$
Y_k = \begin{bmatrix} \frac{s_1}{a} & \frac{s_2}{a} \\ s_1 - a & s_2 - a \end{bmatrix} \begin{bmatrix} s_1^k & 0 \\ 0 & s_2^k \end{bmatrix} C
$$

$$
+ \begin{bmatrix} \frac{s_1}{a} & \frac{s_2}{a} & 1 \\ s_1 - a & s_2 - a & 0 \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{k-1} \begin{bmatrix} s_1^{k-i-1} & 0 \\ 0 & s_2^{k-i-1} \end{bmatrix} P_1 V_i \\ 0 \end{bmatrix}.
$$

Finally, here P_1 is the matrix which contains the right eigenvectors of the finite eigenvalues

$$
P_1 = \begin{bmatrix} a & 1 & \frac{a}{s_1} \\ a & 1 & \frac{a}{s_2} \end{bmatrix}.
$$

Hence,

$$
Y_k = \begin{bmatrix} \frac{s_1}{a} & \frac{s_2}{a} \\ s_1 - a & s_2 - a \end{bmatrix} \begin{bmatrix} s_1^k & 0 \\ 0 & s_2^k \end{bmatrix} C
$$

$$
+ \begin{bmatrix} \frac{s_1}{a} & \frac{s_2}{a} & 1 \\ s_1 - a & s_2 - a & 0 \end{bmatrix} \begin{bmatrix} \sum_{i=0}^{k-1} \begin{bmatrix} s_1^{k-i-1} & 0 \\ 0 & s_2^{k-i-1} \end{bmatrix} \begin{bmatrix} a & 1 & \frac{a}{s_1} \\ a & 1 & \frac{a}{s_2} \end{bmatrix} \begin{bmatrix} G_i \\ 0 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix},
$$

or, equivalently,

$$
Y_k = \begin{bmatrix} s_1^{k+1} c_1 + s_2^{k+1} c_2 + a \sum_{i=0}^{k-1} [(s_1^{k-1} + s_2^{k-1})] G_i \\ a(s_1^k c_1 + s_2^k c_2) + a^2 \sum_{i=0}^{k-1} [(s_1^{k-i-1} + s_2^{k-i-1})] G_i \\ s_1^k (s_1 - a) c_1 + s_2^k (s_2 - a) c_2 + a \sum_{i=0}^{k-1} [((s_1 - a) s_1^{k-1} + (s_2 - a) s_2^{k-1})] G_i \end{bmatrix},
$$

or, equivalently,

$$
\begin{bmatrix} T_k \\ C_k \\ I_k \end{bmatrix} = \begin{bmatrix} s_1^{k+1} c_1 + s_2^{k+1} c_2 + a \sum_{i=0}^{k-1} [(s_1^{k-1} + s_2^{k-1})] G_i \\ a(s_1^k c_1 + s_2^k c_2) + a^2 \sum_{i=0}^{k-1} [(s_1^{k-i-1} + s_2^{k-i-1})] G_i \\ s_1^k (s_1 - a) c_1 + s_2^k (s_2 - a) c_2 + a \sum_{i=0}^{k-1} [((s_1 - a) s_1^{k-1} + (s_2 - a) s_2^{k-1})] G_i \end{bmatrix}.
$$

The proof is completed.

The way this method in this theorem reconstructs the Samuelson's model can be also used in models of similar nature. For example, it can be used into other macroeconomic models, or models where the memory effect appears, and models with delays, see (Dassios et al. 2017; Moaaz et al. 2020a, b). In addition, this updated form of Samuelson's model can provide new alternative methods to prove stability of similar dynamical systems, see (Apostolopoulos and Ortega 2018; Boutarfa and Dassios 2017; Dassios 2015b, 2018a).

4.1 Initial conditions

We assume system (4) and the known initial conditions (IC): Y_2. Note that it is a necessity the initial condition to be Y_2 because Y_k is defined from T_{k-1}, T_{k-2}, and for $k = 2, T_2$ is defined by T_1, T_0.

Definition 4.1   Consider the system (4) with known IC. Then, the IC are called consistent if there exists a solution for the system (4) which satisfies the given conditions.

Proposition 4.1   (See Dassios 2015d) *The IC of system (4) are consistent if and only if*

$$Y_2 \in colspan Q_p + QD_2.$$

Proposition 4.2   (See Dassios et al. 2014) *Consider the system (4) with given IC. Then, the solution for the initial value problem is unique if and only if the IC are consistent. Then, the unique solution is given by the formula*

$$Y_k = Q_p J_p^k Z_2^p + QD_k,$$

where $D_k = \begin{bmatrix} \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i \\ sum_{i=0}^{q_*-1} H_q^i P_2 V_{k+i} \end{bmatrix}$ *and* Z_2^p *is the unique solution of the algebraic system*
$Y_2 = Q_p Z_2^p + D_2.$

Proposition 4.2   *The singular reformulated Samuelson's model (4) has always a unique solution for any given initial conditions.*

Proof   The column Y_k in (4) is defined as

$$Y_k = \begin{bmatrix} T_k \\ C_k \\ I_k \end{bmatrix},$$

whereby for $k = 2$, we get

$$Y_2 = \begin{bmatrix} T_2 \\ C_2 \\ I_2 \end{bmatrix},$$

or, equivalently, using (2), (3)

$$Y_2 = \begin{bmatrix} T_2 \\ aT_1 \\ ab(T_1 - T_0) \end{bmatrix}.$$

or, equivalently,

$$Y_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} T_2 + \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix} aT_1 + \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} aT_0.$$

But $T_2 = a(1 + b)T_1 - abT_0 + \bar{G}$. Thus

$$Y_2 = \begin{bmatrix} 1+b \\ 1 \\ b \end{bmatrix} aT_1 + \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix} aT_0 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bar{G}.$$

However

$$colspanQ_p = \left\langle \begin{bmatrix} 1+b \\ 1 \\ b \end{bmatrix}, \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix} \right\rangle, \quad QD_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence

$$Y_2 \in \left\langle \begin{bmatrix} 1+b \\ 1 \\ b \end{bmatrix}, \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix} \right\rangle + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$Y_2 \in colspanQ_p + QD_2.$$

Hence from Proposition 4.1, the IC of the singular reformulated Samuelson's model (4) are always consistent and from Proposition 4.1, the singular reformulated Samuelson's model (4) has a unique solution for given IC. The proof is completed.

5 Conclusions

In this article, we focused and provided a new alternative formulation of the famous Samuelson macroeconomic model. We proved that this model can be studied via an equivalent singular system of difference equations using pencil theory. We provided properties for existence of solutions. As a future research, it is interesting to study stability and stabilization properties for non-consistent initial conditions. For this case optimization methods, see (Dassios et al. 2015), and concepts from graph theory, see (Cuffe et al. 2016; Dassios et al. 2019), will be required. For this idea, there is already some ongoing research in progress.

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